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Zeros of Difference Polynomials

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Let Δ be the difference operator defined by $\Delta f(x) = f(x+1) - f(x)$. The polynomial $\Delta^m x^n$ of degree $n-m$ is known to have $n-m$ collinear zeros. We study the distribution of these zeros and relate them to zeros of Hermite polynomials. Several open questions are presented.

1. INTRODUCTION

For positive integers m, n, d with

$$0 < m < n, \qquad d = n - m, \tag{1.1}$$

define

$$D(x) = D_{n,m}(x) = \frac{d!}{n!} \Delta^m x^n, \tag{1.2}$$

where Δ is the difference operator defined by

$$\Delta f(x) = f(x+1) - f(x). \tag{1.3}$$

Define the corresponding reduced central difference polynomial $C(z)$ for $d > 1$ by

$$C(z) = C_{n,m}(z) = \begin{cases} i^d D(-i\sqrt{z} - m/2), & \text{if } d \text{ is even} \\ i^d D(-i\sqrt{z} - m/2)/\sqrt{z}, & \text{if } d \text{ is odd.} \end{cases} \tag{1.4}$$

Both $C(z)$ and $D(x)$ are monic polynomials over $\mathbb{Q}$, with

$$d = \deg(D), \qquad c := \deg(C) = [d/2]. \tag{1.5}$$

The study of the zeros of $D(x)$ can be reduced to the study of the zeros of $C(z)$. For, in [1, Theorem 2.3], it is shown that there are c positive

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numbers $y_1 < \cdots < y_c$ such that the d zeros of $D(x)$ are $-m/2 \pm iy_1, ...,$ $-m/2 \pm iy_c$, together with $-m/2$ if d is odd, while $C(z)$ has c distinct positive zeros z_r,

$$0 < z_1 < z_2 < \cdots < z_c, \qquad z_r = y_r^2. \tag{1.6}$$

Call z_c the "spectral radius" of $C(z)$. In [1, Eqs. (1.11), (2.5)], it is noted that

$$m(d-1)/12 \leqslant z_c \leqslant m(d^2 - d)/24 \tag{1.7}$$

for all $d > 1$, and that

$$z_c \sim d^2/(4\pi^2) \text{ as } d \to \infty, \qquad \text{if} \quad m = 1. \tag{1.8}$$

The purpose of this paper is to further study the distribution of zeros of $C(z)$; the primary focus is on the growth of the spectral radius z_c. See Table 5.1 for a list of zeros of $C(z)$ with $4 \leqslant n \leqslant 16$.

In Theorem 2.1, it is shown that for $d > 1$ and any $\varepsilon > 0$,

$$(m/n)^\varepsilon \ll z_c/(nd) \ll (md/n)^\varepsilon; \tag{1.9}$$

i.e., the spectral radius z_c grows much like nd. Some interesting special cases of (1.9) are given at the end of Section 2. For example, if m is bounded, then

$$d^{2-\varepsilon} \ll z_c \ll d^2, \tag{1.10}$$

so the upper bound in (1.7) is sharper than the lower bound for large d. On the other hand, if both d/m and m/d are bounded (e.g., if m/d is constant), then

$$d^2 \ll z_c \ll d^{2+\varepsilon} \tag{1.11}$$

so in this case the lower bound in (1.7) is sharper than the upper bound for large d. The implied constants in (1.9)–(1.11) may depend on ε, but not on m, d. We will sharpen (1.7) and (1.9) for large m; see (1.16).

In Section 3, we make some observations and conjectures on the behavior of the zeros of $C(z)$ based on numerical evidence. For example, we observe that for any fixed integer n, $3 \leqslant n \leqslant 50$, the spectral radius z_c is a unimodal function of the integer m $(1 \leqslant m \leqslant n-2)$, which assumes its maximum at $m = [2n/5]$. If this phenomenon holds for all n, then by (1.11), the "maximum spectral radius function"

$$Z(n) = \max_{1 \leqslant m \leqslant n-2} z_c \tag{1.12}$$

clearly satisfies

$$n^2 \ll Z(n) \ll n^{2+\varepsilon} \tag{1.13}$$

for any $\varepsilon > 0$, where the implied constants may depend on ε (cf. Table 5.2 and Fig. 5.3).

In [1, Theorem 3.2], it is proved that

$$C(z) = \sum_{k=0}^{c} (-1)^k Q_k(m) \binom{d}{2k} z^{c-k}, \tag{1.14}$$

where the polynomial $Q_k(x) \in \mathbb{Q}[x]$ has degree k and is independent of d for each $k \geqslant 0$. While $C(z)$ was originally defined only for positive integers, the definition of $C(z)$ can be extended for all complex m by (1.14).

In general, the zeros of $C(z)$ are not necessarily collinear (or simple). For example, the zeros of $C(z)$ when $d = 15$, $m = 1.1$ are approximately

$$-0.0522163, 1.84607, 6.71539, 0.0368418 \pm 0.152016i, 0.521009 \pm 0.279992i. \tag{1.15}$$

This contrasts with the fact that all zeros of $C(z)$ are positive when m is a positive integer. Theorem 4.3 shows that for all real $m > M(d)$, the zeros of $C(z)$ are again positive, while for all real $m < -M(d)$, the zeros of $C(z)$ are negative. Here $M(d)$ denotes the maximum modulus of the zeros of the polynomial $\mathrm{Disc}(m)$, where $\mathrm{Disc}(m)$ is the discriminant of $C(z)$. The first few values of $M(d)$ are $M(2) = M(3) = 0$ (by convention), $M(4) = 0.2$, $M(5) = 0.6$, $M(6) \sim 1.105031$, $M(7) \sim 1.680194$, $M(8) \sim 2.306474$, $M(9) \sim 2.971947$, $M(10) \sim 3.668542$. We conjecture that for $d \geqslant 4$, $M(d)$ is the absolute value of the leftmost negative zero of $\mathrm{Disc}(m)$. It would be very interesting to analyze the growth of $M(d)$ as $d \to \infty$. It is true that $M(d) < d^2$? Is $M(d)$ monotone increasing?

Theorem 4.1 shows that if $|m| > M(d)$, then the zeros z_ν of $C(z)$ possess convergent expansions of the form $z_\nu = m \sum_{r=0}^{\infty} u_{\nu r} m^{-r}$, where the coefficients $u_{\nu r}$ can be expressed in terms of zeros of Hermite polynomials; see (4.10) and (4.19). We conjecture that $u_{\nu r} = O(d^{r+1})$ as $d \to \infty$; see (4.20). Corollary 4.2 shows, e.g., that for large $|m|$, the spectral radius z_c satisfies

$$\frac{\pi^2}{96}(2d-5)m < z_c < (2d+1)\frac{m}{6}. \tag{1.16}$$

Note that for large $|m|$, (1.16) sharpens (1.9), for each fixed $d > 2$, and (1.16) sharpens (1.7) as well for each fixed $d > 9$.

2. Bounds for the Spectral Radius of $C(z)$

The following theorem shows that the spectral radius z_c of $C(z)$ grows much like nd.

THEOREM 2.1. *Let $1 < d = n - m$. Then for any $\varepsilon > 0$,*

$$(m/n)^{\varepsilon} \ll z_c/(nd) \ll (md/n)^{\varepsilon}, \tag{2.1}$$

where the implied constants may depend on ε (but not on m, d, n).

Proof. Let T_k denote the sum of the kth powers of the zeros of $C(z)$, i.e.,

$$T_k = z_1^k + \cdots + z_c^k. \tag{2.2}$$

In [1, Theorem 4.1], it is shown that for integer $k \geqslant 1$,

$$T_k = \sum_{j=1}^{k} \sum_{i=1}^{2k+1-j} b_{i,j}(k)\, d^i m^j, \tag{2.3}$$

where the coefficients $b_{i,j}(k)$ are rational and satisfy

$$(-1)^{i+j} b_{i,j}(k) < 0. \tag{2.4}$$

Fix an integer $k > 1/\varepsilon$. By (2.3),

$$T_k \ll \sum_{j=1}^{k} d^{2k+1-j} m^j. \tag{2.5}$$

On the other hand, since $b_{2k+1-j,j}(k) > 0$ by (2.4),

$$T_k \gg \sum_{j=1}^{k} d^{2k+1-j} m^j. \tag{2.6}$$

Since $n = m + d$ exceeds both m and d,

$$\sum_{j=1}^{k} d^{2k+1-j} m^j < kmd^{k+1} n^{k-1}. \tag{2.7}$$

Since

$$d^{k-1} + m^{k-1} > ((m+d)/2)^{k-1} = (n/2)^{k-1}, \tag{2.8}$$

$$md^{k+1}(n/2)^{k-1} < (md^{2k} + m^k d^{k+1}) \leqslant \sum_{j=1}^{k} d^{2k+1-j} m^j. \tag{2.9}$$

Combining (2.5) and (2.7), we have

$$T_k \ll (nd)^k \binom{md}{n}. \tag{2.10}$$

Combining (2.6) and (2.9), we have

$$(nd)^k\,(m/n) \ll T_k/c. \tag{2.11}$$

Because the zeros of $C(z)$ are positive, it follows from (2.2) that

$$(T_k/c)^{1/k} \leqslant z_c \leqslant T_k^{1/k}. \tag{2.12}$$

By (2.10)–(2.12),

$$nd(m/n)^{1/k} \ll z_c \ll nd(md/n)^{1/k}. \tag{2.13}$$

Since $\varepsilon > 1/k$, $(m/n)^{1/k} > (m/n)^{\varepsilon}$. Also, $(2md/n)^{1/k} < (2md/n)^{\varepsilon}$, because $2md > n$. Thus the result follows from (2.13).

If m is bounded, then Theorem 2.1 yields

$$d^{2-\varepsilon} \ll z_c \ll d^2. \tag{2.14}$$

(This is consistent with (1.8).) In this case, the upper bound in (1.7) is sharper than the lower bound, for large d.

If d is bounded, Theorem 2.1 yields

$$m \ll z_c \ll m. \tag{2.15}$$

(This is consistent with (1.7).)

If m/d and d/m are both bounded, Theorem 2.1 yields

$$d^2 \ll z_c \ll d^{2+\varepsilon}. \tag{2.16}$$

In this case, the lower bound in (1.7) is sharper than the upper bound, for large d.

If m/d tends to zero, Theorem 2.1 yields

$$d^{2-\varepsilon} \ll z_c \ll d^{2+\varepsilon}. \tag{2.17}$$

If d/m tends to zero, Theorem 2.1 yields

$$md \ll z_c \ll md^{1+\varepsilon}. \tag{2.18}$$

3. Conjectures and Observations on Zeros of $C(z)$

The spectral radius z_c has been defined as the largest zero of $C(z) = C_{n,m}(z)$. More generally, for a nonnegative integer k, define $z_{c-k}(n, m)$ to be the $(k+1)$st largest zero of $C(z)$. Note that z_{c-k} is meaningful only if $c > k$, i.e., if $n - m = d \geqslant 2k + 2$. In particular, $n \geqslant 2k + 3$.

Conjecture 3.1. For fixed integers n and k, z_{c-k} is a unimodal function of the integer m, $1 \leqslant m \leqslant n - 2k - 2$. In particular, the spectral radius is unimodal for $1 \leqslant m \leqslant n - 2$.

We have verified Conjecture 3.1 for all $n \leqslant 50$. In the case $k = 0$, we were surprised to see that for each fixed $n \leqslant 50$, the peak of the unimodal function z_c always occurs at

$$m = [2n/5]. \tag{3.1}$$

More generally, for $0 \leqslant k \leqslant 9$, $n \leqslant 50$, the peak of the unimodal function z_{c-k} always occurs at

$$m = [(2n - 3k)/5], \tag{3.2}$$

provided that $n \geqslant n_1(k)$, where

$$\begin{gathered} n_1(0) = 3, n_1(1) = 5, n_1(2) = 9, n_1(3) = 13, n_1(4) = 17, n_1(5) = 26, \\ n_1(6) = 30, n_1(7) = 39, n_1(8) = 43, n_1(9) = 47. \end{gathered} \tag{3.3}$$

For small values of $n < n_1(k)$, another curiously regular phenomenon was observed. Namely, for $0 \leqslant k \leqslant 10$, $n \leqslant 50$, the peak of the unimodal function z_{c-k} always occurred at

$$m = [(n - 2k - 1)/2], \tag{3.4}$$

provided that $n \leqslant n_0(k)$, where

$$\begin{gathered} n_0(0) = 6, n_0(1) = 10, n_0(2) = 14, n_0(3) = 18, n_0(4) = 22, n_0(5) = 24, \\ n_0(6) = 28, n_0(7) = 30, n_0(8) = 34, n_0(9) = 38, n_0(10) = 40. \end{gathered} \tag{3.5}$$

Define the "maximum spectral radius" function $Z(n)$ by

$$Z(n) = \max_{1 \leqslant m \leqslant n-2} z_c. \tag{3.6}$$

See Table 5.2 for a list of values of $Z(n)$, $4 \leqslant n \leqslant 50$.

Conjecture 3.2. For all $\varepsilon > 0$,

$$n^2 \ll Z(n) \ll n^{2+\varepsilon}, \qquad n \to \infty.$$

If the spectral radius z_c always assumes its maximum at $m = [2n/5]$ (see (3.1)), then Conjecture 3.2 is valid by (2.16). Suggestive evidence supporting Conjecture 3.2 is provided by Fig. 5.3.

A possible approach to settling the conjectures in this section is to consider z_c as a continuous function of real (rather than integer) variables m, n. Despite the considerable amount of literature on fractional finite difference operators, we have been unable to find the appropriate definition of $C(z)$ as a function of real m, n.

4. Expansions of Zeros of $C(z)$ in Descending Powers of m

Throughout this section, d is an *integer* >1. In view of (1.14), the polynomial $C(z)$ is well-defined for all complex m. We will drop the restriction that m be an integer and allow m to be complex in this section.

Let $h_1, ..., h_c$ denote the c positive zeros of the dth Hermite polynomial

$$H_d = H_d(x) = \sum_{k=0}^{c} (-1)^k \frac{d!}{(d-2k)!\,k!} (2x)^{d-2k} \tag{4.1}$$

with

$$0 < h_1 < h_2 < \cdots < h_c; \tag{4.2}$$

see [4, pp. 106, 130]. The d zeros of H_d are $\pm h_\nu$ $(1 \leqslant \nu \leqslant c)$ together with 0 if d is odd. We will relate the zeros of $C(z)$ to the zeros h_ν of H_d in Theorem 4.1.

With the polynomials $Q_k(x)$ of degree k appearing in (1.14), define polynomials $F(w, v) = F_d(w, v)$ by

$$F(w, v) = \sum_{k=0}^{c} (-1)^k \binom{d}{2k} Q_k(w^{-1}) w^k v^{d-2k}. \tag{4.3}$$

If for a fixed complex $w = w_0$, the d zeros v_i of $F(w_0, v)$ are distinct, then by a classical version of the implicit function theorem [3, p. 170; 2, p. 105], there are d analytic functions $v_i(w)$ in a neighborhood of $w = w_0$ such that

$$0 = F(w, v_i(w)), \qquad v_i(w_0) = v_i. \tag{4.4}$$

Since $Q_k(x)$ has leading term $(2k)!\,(x/24)^k/k!$ by [1, Eq. (3.17)], it follows from (4.3) and (4.1) that

$$F(0, v) = \sum_{k=0}^{c} (-1)^k \binom{d}{2k} \frac{(2k)!}{k!\,24^k} v^{d-2k} = 24^{-d/2} H_d(v\sqrt{6}). \tag{4.5}$$

Since the d zeros of H_d are distinct, there are c analytic functions $v_\nu(w)$ $(1 \leqslant \nu \leqslant c)$ in a neighborhood of $w=0$ such that

$$0 = F(w, v_\nu(w)), \qquad v_\nu(0) = h_\nu/\sqrt{6}. \tag{4.6}$$

We proceed to extend the local functions $v_\nu(w)$ to global ones.

For each d, the discriminant of $C(z)$ is a polynomial in m over $\mathbb{Q}$, by (1.14). Let $M(d)$ denote the maximum modulus of the zeros of this discriminant polynomial in m. If $|m| > M(d)$, the c zeros of $C(z)$ are distinct. Thus, for each fixed complex w with $|w| < M(d)^{-1}$, the zeros v_i of $F(w, v)$ are distinct, since

$$m^c F(1/m, \sqrt{z/m}) = \begin{cases} C(z) & \text{if } \ d \text{ is even} \\ \sqrt{z/m}\, C(z), & \text{if } \ d \text{ is odd,} \end{cases} \tag{4.7}$$

by (4.3) and (1.14). Since the disk $|w| < M(d)^{-1}$ is simply connected, it follows from the monodromy theorem that there exist c analytic functions $v_\nu(w)$ $(1 \leqslant \nu \leqslant c)$ on the *entire disk* $|w| < M(d)^{-1}$ such that (4.6) holds. In view of (4.7), the c zeros of the polynomial $C(z)$ are given by

$$z_\nu = m v_\nu(1/m)^2, \qquad 1 \leqslant \nu \leqslant c, \tag{4.8}$$

when $|m| > M(d)$.

THEOREM 4.1. *Assume that $|m| > M(d)$. Then the zeros z_ν $(1 \leqslant \nu \leqslant c)$ of $C(z)$ have the convergent expansions*

$$z_\nu = \sum_{r=0}^{\infty} u_{\nu r} m^{1-r}, \tag{4.9}$$

where

$$u_{\nu 0} = h_\nu^2/6 \tag{4.10}$$

and where for each pair r, d, there is a polynomial $f_{r,d}$ in $\mathbb{Q}[x]$ such that

$$u_{\nu r} = f_{r,d}(u_{\nu 0}), \qquad 1 \leqslant \nu \leqslant c. \tag{4.11}$$

Proof. For $1 \leqslant \nu \leqslant c$, write

$$v_\nu(w) = \sum_{r=0}^{\infty} \frac{v_{\nu t} w^r}{r!}, \qquad |w| < M(d)^{-1}. \tag{4.12}$$

From (4.6), we have

$$v_{\nu 0} = h_\nu/\sqrt{6} \tag{4.13}$$

and

$$0 = F(w, v_\nu(w)) = \sum_{k=0}^{c} (-1)^k \binom{d}{2k} Q_k(w^{-1})\, w^k \left(\sum_{r=0}^{\infty} \frac{v_{\nu t} w^r}{r!} \right)^{d-2k} \tag{4.14}$$

Fix $t \geqslant 1$. The coefficient of w^t in the Taylor series expansion of the right member of (4.14) equals a polynomial in $v_{\nu 0}, v_{\nu 1}, \ldots, v_{\nu, t-1}$ plus

$$\frac{v_{\nu t}}{t!} \sum_{k=0}^{c} (-1)^k \binom{d}{2k} \frac{(2k)!}{k!\, 24^k} (d-2k) v_{\nu 0}^{d-2k-1}. \tag{4.15}$$

By (4.5), the expression in (4.15) equals

$$\frac{v_{\nu t}}{t!} 24^{-d/2} \sqrt{6}\, H_d'(v_{\nu 0} \sqrt{6}), \tag{4.16}$$

which is a nonzero multiple of $v_{\nu t}$ by (4.13). Since the coefficient of w^t in (4.14) vanishes, it follows by induction on t that

$$v_{\nu r} = g_{r,d}(v_{\nu 0}), \qquad 1 \leqslant \nu \leqslant c, \tag{4.17}$$

where $g_{r,d}$ is a polynomial over $\mathbb{Q}$ with all exponents of the same parity (note that $\mathbb{Q}[v_{\nu 0}^2] = \mathbb{Q}(v_{\nu 0}^2)$. The result now follows from (4.12), (4.13), (4.17), and (4.8).

By a more involved argument (which we omit), it can be shown that for each $r \geqslant 0$, there exist polynomials $f_r(x, y)$ and $g_r(x, y)$ in $\mathbb{Q}[x, y]$ such that

$$v_{\nu r} = v_{\nu 0}\, g_r(v_{\nu 0}^2, d) \qquad (1 \leqslant \nu \leqslant c) \tag{4.18}$$

and

$$u_{\nu r} = u_{\nu 0}\, f_r(u_{\nu 0}, d) \qquad (1 \leqslant \nu \leqslant c). \tag{4.19}$$

Tables 5.4 and 5.5 give these polynomials for $r = 1, 2, 3$. These tables suggest the conjecture that both f_r and g_r have total degree r. If this conjecture is true, then by (4.19),

$$u_{\nu r} = O(d^{r+1}) \qquad \text{as} \quad d \to \infty, \tag{4.20}$$

for each fixed pair ν, r, since the zeros of H_d are $O(d^{1/2})$ [4, p. 130, Eq. (6.31.19)].

By Theorem 4.1, z_ν behaves approximately like a linear function of m for large $|m|$, namely

$$u_{\nu 0} m + u_{\nu 1} = h_\nu^2 m/6 + h_\nu^2 (2h_\nu^2 - 2d + 3)/60; \tag{4.21}$$

see Table 5.5. For an example of use of Theorem 4.1 for numerical approximation of the smallest positive zero of $C(z)$, let $m = 6$, $d = 10$, $\nu = 1$.

Then $h_1 = 0.342901327...$, $m > M(d) = 3.6...$, and $z_1 = 0.088104...$ can be approximated by $mu_{10} + u_{11} + u_{12}/m + u_{13}/m^2 = 0.088110....$

The following result sharpens Theorem 2.1 for large $|m|$ and fixed $d > 1$.

COROLLARY 4.2. *For large $|m|$ and fixed $d > 1$,*

$$\frac{m\pi^2(4v-1-(-1)^d)^2}{96(2d+1)} < |z_v| < \frac{m(4v+2-(-1)^d)^2}{6(2d+1)}, \qquad 1 \leqslant v \leqslant c. \tag{4.22}$$

Proof. This result follows immediately from Theorem 4.1 and the following bound for positive zeros of Hermite polynomials [4, p. 130, Eq. (6.31.19)]:

$$\frac{\pi(4v-1-(-1)^d)}{4\sqrt{2d+1}} < h_v < \frac{4v+2-(-1)^d}{\sqrt{2d+1}} \qquad (1 \leqslant v \leqslant c). \tag{4.23}$$

THEOREM 4.3. *Let $|m| > M(d)$ with m real. Then each zero z_v of $C(z)$ is real and has the same sign as m. In particular, if $m < -M(d)$, then the d zeros of $D(x)$ are all real.*

Proof. By (4.12), (4.13), and (4.17), $v_v(w)$ is real. Thus z_v has the same sign as m by (4.8). Finally, if $m < -M(d)$, then the zeros of $C(z)$ are negative, so the zeros of $D(x)$ are real by (1.4).

5. TABLES AND GRAPH

TABLE 5.1

Zeros of $C_{n,m}(z)$

$n=4$	$m=1$	0.250000		
$n=4$	$m=2$	0.166667		
$n=5$	$m=1$	0.473607	0.026393	
$n=5$	$m=2$	0.500000		
$n=5$	$m=3$	0.250000		
$n=6$	$m=1$	0.750000	0.083333	
$n=6$	$m=2$	0.928174	0.071826	
$n=6$	$m=3$	0.750000		
$n=6$	$m=4$	0.333333		
$n=7$	$m=1$	1.077985	0.158991	0.013024
$n=7$	$m=2$	1.434259	0.232408	
$n=7$	$m=3$	1.382456	0.117544	
$n=7$	$m=4$	1.000000		
$n=7$	$m=5$	0.416667		

(Table continued)

TABLE 5.1—*Continued*

$n=8$	$m=1$	1.457107	0.250000	0.042893			
$n=8$	$m=2$	2.011743	0.448691	0.039566			
$n=8$	$m=3$	2.116025	0.383975				
$n=8$	$m=4$	1.836660	0.163340				
$n=8$	$m=5$	1.250000					
$n=8$	$m=6$	0.500000					
$n=9$	$m=1$	1.887158	0.355069	0.083333	0.007773		
$n=9$	$m=2$	2.657329	0.710202	0.132468			
$n=9$	$m=3$	2.938034	0.741999	0.069967			
$n=9$	$m=4$	2.797055	0.536278				
$n=9$	$m=5$	2.290833	0.209167				
$n=9$	$m=6$	1.500000					
$n=9$	$m=7$	0.583333					
$n=10$	$m=1$	2.368034	0.473607	0.131966	0.026393		
$n=10$	$m=2$	3.369018	1.012995	0.259568	0.025086		
$n=10$	$m=3$	3.841976	1.170624	0.237400			
$n=10$	$m=4$	3.861914	1.037019	0.101067			
$n=10$	$m=5$	3.477767	0.688900				
$n=10$	$m=6$	2.744990	0.255010				
$n=10$	$m=7$	1.750000					
$n=10$	$m=8$	0.666667					
$n=11$	$m=1$	2.899676	0.605308	0.187708	0.052140	0.005168	
$n=11$	$m=2$	4.145446	1.355514	0.412823	0.086217		
$n=11$	$m=3$	4.823945	1.660554	0.468988	0.046513		
$n=11$	$m=4$	5.021271	1.633240	0.345489			
$n=11$	$m=5$	4.784706	1.332875	0.132420			
$n=11$	$m=6$	4.158312	0.841688				
$n=11$	$m=7$	3.199138	0.300862				
$n=11$	$m=8$	2.000000					
$n=11$	$m=9$	0.750000					
$n=12$	$m=1$	3.482051	0.750000	0.250000	0.083333	0.017949	
$n=12$	$m=2$	4.985606	1.737222	0.588266	0.171574	0.017332	
$n=12$	$m=3$	5.881319	2.207146	0.750000	0.161535		
$n=12$	$m=4$	6.269176	2.309628	0.685082	0.069447		
$n=12$	$m=5$	6.198061	2.097151	0.454788			
$n=12$	$m=6$	5.706914	1.629195	0.163891			
$n=12$	$m=7$	4.838760	0.994573				
$n=12$	$m=8$	3.653280	0.346720				
$n=12$	$m=9$	2.250000					
$n=12$	$m=10$	0.833333					
$n=13$	$m=1$	4.115136	0.907581	0.318529	0.119111	0.035958	0.003686
$n=13$	$m=2$	5.888706	2.158048	0.783700	0.275462	0.060751	
$n=13$	$m=3$	7.012192	2.807886	1.073129	0.323620	0.033172	

(Table continued)

TABLE 5.1—*Continued*

$n=13$	$m=4$	7.601686	3.057997	1.097241	0.243076			
$n=13$	$m=5$	7.709731	2.960089	0.903913	0.092934			
$n=13$	$m=6$	7.373471	2.561848	0.564681				
$n=13$	$m=7$	6.628771	1.925800	0.195430				
$n=13$	$m=8$	5.519146	1.147520					
$n=13$	$m=9$	4.107418	0.392582					
$n=13$	$m=10$	2.500000						
$n=13$	$m=11$	0.916667						
$n=14$	$m=1$	4.798917	1.077985	0.393104	0.158991	0.057979	0.013024	
$n=14$	$m=2$	6.854100	2.618145	0.997788	0.394831	0.122441	0.012695	
$n=14$	$m=3$	8.215098	3.461332	1.434489	0.521867	0.117214		
$n=14$	$m=4$	9.015982	3.873567	1.570141	0.489509	0.050801		
$n=14$	$m=5$	9.314394	3.909712	1.448891	0.327004			
$n=14$	$m=6$	9.147658	3.611515	1.124146	0.116681			
$n=14$	$m=7$	8.548038	3.027052	0.674910				
$n=14$	$m=8$	7.550401	2.222591	0.227008				
$n=14$	$m=9$	6.199490	1.300510					
$n=14$	$m=10$	4.561553	0.438447					
$n=14$	$m=11$	2.750000						
$n=14$	$m=12$	1.000000						
$n=15$	$m=1$	5.533386	1.261170	0.473607	0.202682	0.083333	0.026393	0.002762
$n=15$	$m=2$	7.881249	3.117772	1.229655	0.527915	0.198228	0.045180	
$n=15$	$m=3$	9.488859	4.166621	1.831942	0.750183	0.237540	0.024856	
$n=15$	$m=4$	10.509929	4.753351	2.097116	0.792206	0.180730		
$n=15$	$m=5$	11.008284	4.938802	2.072933	0.660731	0.069250		
$n=15$	$m=6$	11.022719	4.762285	1.802912	0.412084			
$n=15$	$m=7$	10.583954	4.263612	1.345198	0.140570			
$n=15$	$m=8$	9.722053	3.492599	0.785348				
$n=15$	$m=9$	8.471876	2.519511	0.258613				
$n=15$	$m=10$	6.879803	1.453530					
$n=15$	$m=11$	5.015686	0.484314					
$n=15$	$m=12$	3.000000						
$n=15$	$m=13$	1.083333						
$n=16$	$m=1$	6.318536	1.457107	0.559957	0.250000	0.111616	0.042893	0.009892
$n=16$	$m=2$	8.969684	3.657236	1.478693	0.673632	0.285702	0.092019	0.009701
$n=16$	$m=3$	10.832496	4.923225	2.264313	1.004899	0.386044	0.089024	
$n=16$	$m=4$	12.081836	5.695377	2.674211	1.141740	0.368030	0.038806	
$n=16$	$m=5$	12.788584	6.042709	2.766207	1.071526	0.247641		
$n=16$	$m=6$	12.993913	6.004386	2.579006	0.834592	0.088104		
$n=16$	$m=7$	12.728290	5.615541	2.158361	0.497808			
$n=16$	$m=8$	12.019168	4.916185	1.566767	0.164546			
$n=16$	$m=9$	10.895685	3.958391	0.895923				
$n=16$	$m=10$	9.393240	2.816524	0.290236				
$n=16$	$m=11$	7.560095	1.606571					
$n=16$	$m=12$	5.469818	0.530182					
$n=16$	$m=13$	3.250000						
$n=16$	$m=14$	1.166667						

TABLE 5.2

Table of $Z(n)$—Maximum Spectral Radius Function [see (3.7)]

4	0.250000	28	49.382763
5	0.500000	29	53.575608
6	0.928174	30	57.873439
7	1.434258	31	62.466259
8	2.116025	32	67.155896
9	2.938033	33	72.056263
10	3.861913	34	77.149253
11	5.021271	35	82.349257
12	6.269176	36	87.847622
13	7.709731	37	93.443404
14	9.314394	38	99.254114
15	11.022719	39	105.257903
16	12.993912	40	111.371572
17	15.057720	41	117.784952
18	17.320743	42	124.296196
19	19.761363	43	131.027035
20	22.305547	44	137.949990
21	25.128667	45	144.986444
22	28.046498	46	152.322389
23	31.167635	47	159.756542
24	34.474462	48	167.415428
25	37.885140	49	175.264322
26	41.584721	50	183.231006
27	45.380277		

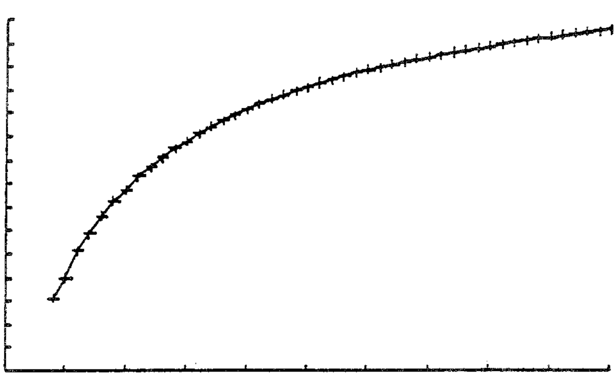

FIG. 5.3. Graph of $y = Z(n)/n^2$ (n-axis from 0 to 50 with tic marks at 5; y-axis from 0 to 0.075 with tic marks at 0.005).

TABLE 5.4

Table of $v_k = v_{vk}$ [See (4.12)]

$$v_1 = \frac{3v_0^3}{5} - \frac{dv_0}{10} + \frac{3v_0}{20}$$

$$v_2 = \frac{87v_0^5}{175} + \frac{8dv_0^3}{175} + \frac{22v_0^3}{175} + \frac{d^2v_0}{140} - \frac{39dv_0}{700} + \frac{191v_0}{2800}$$

$$v_3 = \frac{81v_0^7}{125} + \frac{1107dv_0^5}{1750} - \frac{8289v_0^5}{3500} - \frac{657d^2v_0^3}{3500} + \frac{27dv_0^3}{28} - \frac{14283v_0^3}{14000} + \frac{39d^3v_0}{7000} - \frac{99d^2v_0}{2800} + \frac{201dv_0}{4000} - \frac{531v_0}{56000}$$

TABLE 5.5

Table of $u_k = u_{vk}$ [See (4.9)]

$$u_1 = \frac{6u_0^2}{5} - \frac{du_0}{5} + \frac{3u_0}{10}$$

$$u_2 = -\frac{24u_0^3}{175} - \frac{13du_0^2}{175} + \frac{107u_0^2}{350} + \frac{3d^2u_0}{175} - \frac{3du_0}{35} + \frac{127u_0}{1400}$$

$$u_3 = -\frac{72u_0^4}{875} + \frac{36du_0^3}{125} - \frac{138u_0^3}{175} - \frac{11d^2u_0^2}{175} + \frac{247du_0^2}{875} - \frac{981u_0^2}{3500} + \frac{d^3u_0}{875} - \frac{9d^2u_0}{1750} + \frac{11du_0}{7000} + \frac{99u_0}{14000}$$

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